Advanced Systems Theory

09/04/2026, Thursday, 8:30 – 10:30

1 Disturbance decoupling with measurement feedback (15 + 10 = 25 pts)

Consider the system

$$\dot{x} = Ax + Bu + Ed$$
$$y = Cx$$
$$z = Hx$$

with

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad E = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, H = \begin{bmatrix} 0 & 1 & a \end{bmatrix}$$

where a is a real number.

(a) Determine all values of a such that the problem of disturbance decoupling (from d to z) by measurement feedback is solvable.

(b) For those values of a, compute a dynamic controller that makes the system from d to z decoupled.

2 Lyapunov equation (10 + 10 = 20 pts)

Let $Q = [q_{ij}]$ be an $n \times n$ real symmetric matrix and let d_i with $1 \leqslant i \leqslant n$ be positive numbers. Define $P = [p_{ij}]$ by

$$p_{ij} = \frac{q_{ij}}{d_i + d_j}$$

where $1 \leqslant i, j \leqslant n$.

(a) Find an $n \times n$ diagonal matrix D such that $DP + PD = Q$.

(b) Show that P is positive semidefinite if so is Q.

3 Synchronization (25 pts)

Consider the multi-agent system given by the communication graph G agent dynamics

$$\dot{x}_i = Ax_i + Bz_i$$

and the diffusive coupling

$$z_i = K \sum_{j \in N(i)} (x_j - x_i)$$

where $i \in \{1, 2, \ldots, p\}$. Suppose that the graph G is connected and the smallest nonzero eigenvalue of its Laplacian is λ_2. Find K such that the multi-agent system is synchronized by solving the algebraic Riccati equation (via Hamiltonian)

$$A^T P + PA - 2\lambda_2 PBB^T P + Q = 0$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \lambda_2 = \frac{1}{2}, \quad \text{and} \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

4 Data-driven control (20 pts)

Consider the true unknown system

$$\boldsymbol{x}(t+1) = A_{\text{true}}\boldsymbol{x}(t) + B_{\text{true}}\boldsymbol{u}(t)$$

where $t \in \mathbb{Z}_+$, $\boldsymbol{x} \in \mathbb{R}^n$, and $\boldsymbol{u} \in \mathbb{R}^m$. Suppose that the data

$$U_- = \begin{bmatrix} u(0) & u(1) & \cdots & u(T-1) \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} x(0) & x(1) & \cdots & x(T) \end{bmatrix}$$

are harvested from the true system. Define

$$X_- := \begin{bmatrix} x(0) & x(1) & \cdots & x(T-1) \end{bmatrix} \quad \text{and} \quad X_+ := \begin{bmatrix} x(1) & x(1) & \cdots & x(T) \end{bmatrix}.$$

Also, define the set of all explaining systems

$$\Sigma := \{(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \mid X_+ = AX_- + BU_-\}.$$

The data (U_-, X) are said to be *informative for stability* if A is stable for all A such that $(A, B) \in \Sigma$ for some B. Show that the data (U_-, X) are informative for stability if and only if X_- has full row rank and there exists a right inverse $X_-^\dagger$ of X_- such that $X_+X_-^\dagger$ is stable and $U_-X_-^\dagger = 0$.

HINT: For the 'only if' part, try first to show that X_- has full row rank and

$$\ker \begin{bmatrix} X_-^T & U_-^T \end{bmatrix} \subseteq \ker \begin{bmatrix} I_n & 0_{n,m} \end{bmatrix}.$$

10 pts free